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THE EQUIVARIANT SIGNATURE OF HYPERSURFACE SINGULARITIES AND ETA-INVARIANT

ANDRÁS NÉMETHI†

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1. INTRODUCTION

Let f be an analytic germ of a hypersurface singularity. The associated monodromy action preserves the intersection form of its Milnor fibre. In this paper we study the corresponding equivariant signatures. The guiding principle is the following. Write f as a composed singularity $f = p \circ \phi$, where ϕ (resp. p) is an isolated complete intersection (resp. curve) singularity; express the equivariant signature of f as the equivariant signature of p with coefficients in a (non-degenerate) hermitian flat bundle; identify this with an index of the signature bundle; and when the primary invariant (Chern class) vanishes, express it in terms of the eta-invariant of the boundary (of the Milnor fiber of p) with coefficients in the corresponding signature bundle. In the realization of this program, we have two basic obstructions: The monodromy action of p (resp. f) is not compact (finite), and the monodromy representation of ϕ (i.e. the candidate for the flat bundle), in general, is degenerate. This second obstruction is solved by introducing the variation map of ϕ . The variation structure (i.e. the degenerate monodromy representation together with the variation map) substitutes perfectly the non-degenerate representations. It turns out that the equivariant signature can be expressed in terms of this variation structure and the geometry of the curve singularity p . It has a sum decomposition, corresponding to the Jaco–Shalen–Johansson (or splice) decomposition of the link complement of $p^{-1}(0)$; each term is closely related to the Seifert geometry of the components. (The structure of the fundamental group of the Seifert components will remove the first obstruction too.) The primary invariant (here the first Chern class) vanishes when the variation structure is abelian or the intersection form of ϕ is definite. The basic application for the first case is the topological series; the coverings exemplify the second case.

The equivariant signature has been computed only for a few families of isolated singularities: curve singularities [15], quasi-homogeneous germs [20], suspensions [13]. This paper gives the variable term of the composed topological series (in particular for the Yomdin's series) (see Corollaries 5.4 and 5.11), and reduces the general case (see 5.1) to a signature computation of a non-degenerate hermitian flat bundle over the r -punctured 2-dimensional sphere (for which there exists a clear algorithm [8]).

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Current address: Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, OH 43210-1174, U.S.A. E-mail: nemethi@math.ohio-state.edu.

The paper is organized as follows. Section 2 contains the definition of the isometric and variation structures and our eta-invariants associated with them. In the next section we state our general results about the (equivariant) signature of hypersurface singularities. Their proofs can be found in section 4. The last section contains the applications, and some connections with the limit mixed Hodge structures.

2. THE ETA-INVARIANT OF THE VARIATION STRUCTURES

2.1. If U is a finite-dimensional vector space then U^* is its dual $\text{Hom}_{\mathbb{C}}(U, \mathbb{C})$. There is a natural isomorphism $\theta: U \rightarrow U^{**}$ given by $\theta(u)(\varphi) = \varphi(u)$. We denote by $\bar{\cdot}$ the complex conjugation. If $\varphi \in \text{Hom}_{\mathbb{C}}(U, U')$, then $\bar{\varphi} \in \text{Hom}_{\mathbb{C}}(U, U')$ is defined by $\bar{\varphi}(x) := \overline{\varphi(\bar{x})}$, and the dual $\varphi^*: U'^* \rightarrow U^*$ of φ by $\varphi^*(\psi) = \psi \circ \varphi$.

A \mathbb{C} -linear endomorphism $b: U \rightarrow U^*$ with $\overline{b^* \circ \theta} = \varepsilon b$ ($\varepsilon = \pm 1$) is called ε -hermitian form on U . The automorphisms $h: U \rightarrow U$ with $\bar{h}^* \circ b \circ h = b$ form the orthogonal group $\text{Aut}(U; b)$. Any $h \in \text{Aut}(U; b)$ determines a spectral decomposition $(U; b, h) = \oplus_{\chi} (U_{\chi}; b_{\chi}, h_{\chi})$, where U_{χ} is the generalized χ -eigenspace of h .

Definition 2.2. The eta-invariant $\eta_b(h)$ is defined by the sum $\sum_{\chi} \eta_b(h)_{\chi}$, where

$$\eta_b(h)_{\chi} = \begin{cases} (1 - 2c) \text{sign } b_{\chi} & \text{if } \chi = e^{2\pi ic}, 0 < c < 1 \\ -\varepsilon \text{sign}[b_{\chi}(h_{\chi} - h_{\chi}^{-1})] & \text{if } \chi = 1. \end{cases}$$

By convention, the signature of a (-1) -hermitian form b is $\text{sign } b := \text{sign}(ib)$. In all our cases χ lies on the unit circle S^1 . For the sake of exactness, we can define $\eta_{\chi} = 0$ provided that $\chi \notin S^1$.

Notice also the similarity with the function $2((\cdot)) : \mathbb{R} \rightarrow \mathbb{R}$, used in number theory, defined by $2((c)) = 0$ if $c \in \mathbb{Z}$, and $2((c)) = 1 - 2\{c\}$ if $c \notin \mathbb{Z}$ ($\{\cdot\}$ denotes the fractional part).

The above eta-invariant is a “good object” only for non-degenerate hermitian forms b (i.e. when b is an isomorphism). In that case η_b can be interpreted as the η -invariant of the signature operator of the circle twisted with the signature bundle of a hermitian flat bundle, in the sense of [2] (cf. (3.4) and Lemma 4.11).

Definition 2.3. An ε -hermitian isometric structure of the group G is a system $\mathcal{I} = (U; b, \rho)$ such that b is an ε -hermitian non-degenerate form, and $\rho: G \rightarrow \text{Aut}(U; b)$ is a group endomorphism. Given a system \mathcal{I} and an element $g \in G$, we define the eta-invariant $\eta_{\mathcal{I}}(g)$ by $\eta_b(\rho(g))$.

2.4. Any representation $\rho: G \rightarrow \text{Aut}(U)$ defines a left action of G on $\text{Hom}(U^*, U)$ by $g * \varphi = \rho(g) \circ \varphi$. Then, by definition, a twisted-homeomorphism is a map $V: G \rightarrow \text{Hom}(U^*, U)$ with $V(gh) = \rho(g) \circ V(h) + V(g)$.

Definition 2.5. An ε -hermitian variation structure of the group G is a system $\mathcal{V} = (U; b, \rho, V)$ such that b is an ε -hermitian (maybe degenerate) form, ρ is a representation of G in $\text{Aut}(U; b)$, V is a twisted-homeomorphism with respect to the left action of G via ρ , and they satisfy the following compatibility conditions for any $g \in G$:

- (i) $\overline{\theta^{-1} \circ V(g)}^* = -\varepsilon V(g) \circ \overline{\rho(g)}^*$, and
- (ii) $V(g) \circ b = \rho(g) - I$.

In the sequel we identify U^{**} and U by the map θ .

Notice that if b is non-degenerate, then the variation structure \mathcal{V} is completely determined by the underlying isometric structure $\mathcal{J} = (U; b, \rho)$ (use (ii)). As we will see, the category of variation structures substitutes perfectly the category of isometric structures in those cases when the geometric situation gives a degenerate hermitian form.

Definition 2.6. Let \mathcal{V} be an ε -hermitian variation structure. Then for any $g \in G$, $\rho(g)$ determines a spectral decomposition $(U; b, \rho(g), V(g)) = \oplus_{\chi}(U_{\chi}; b_{\chi}, \rho(g)_{\chi}, V(g)_{\chi})$, where U_{χ} is the generalized χ -eigenspace of $\rho(g)$.

The eta-invariant $\eta_{\mathcal{V}}(g)$ is defined by the sum $\sum_{\chi} \eta_{\mathcal{V}}(g)_{\chi}$, where

$$\eta_{\mathcal{V}}(g)_{\chi} = \begin{cases} (1 - 2c) \operatorname{sign} b_{\chi} & \text{if } \chi = e^{2\pi i c}, 0 < c < 1 \\ -\operatorname{sign}[(1 + \rho(g)_{\chi}^{-1})V(g)_{\chi}] & \text{if } \chi = 1. \end{cases}$$

(In the last section we will use the notation $\eta(\mathcal{V}; g) = \eta_{\mathcal{V}}(g)$ too.)

Remark 2.7. If \mathcal{V} is a variation structure with non-degenerate b , and \mathcal{J} is the underlying isometric structure, then for any g one has: $\eta_{\mathcal{V}}(g) = \eta_{\mathcal{J}}(g)$. To see this, use Definition 2.5(ii) and the fact that for any non-degenerate b and $h \in \operatorname{Aut}(U; b)$ one has $\operatorname{sign}[b(h - h^{-1})] = \varepsilon \operatorname{sign}[(h - h^{-1})b^{-1}]$.

LEMMA 2.8. Let b be an ε -hermitian form, $\rho: G \rightarrow \operatorname{Aut}(U; b)$ a representation and $V: G \rightarrow \operatorname{Hom}_{\mathbb{C}}(U^*, U)$ a map. Define $U^e := U^* \oplus U$, the map $\rho^e: G \rightarrow \operatorname{Aut}(U^e)$ by

$$\rho^e(g) = \begin{pmatrix} 1 & 0 \\ V(g) & \rho(g) \end{pmatrix} \quad \text{and} \quad b^e = \begin{pmatrix} 0 & \varepsilon \\ 1 & b \end{pmatrix}.$$

Then $\mathcal{V} = (U; b, \rho, V)$ is an ε -hermitian variation structure if and only if $\mathcal{J}^e = (U^e; b^e, \rho^e)$ is an ε -hermitian isometric structure. Moreover, for any $g \in G$ one has

$$\eta_{\mathcal{V}}(g) = \eta_{\mathcal{J}^e}(g).$$

The proof is left to the reader.

2.9. In the computation of the equivariant signature we will need a more complicated invariant too.

Let \mathcal{J} be an isomorphic structure of the group G , $\lambda \in S^1$, $m \in \mathbb{N}^*$ and $o, g \in G$ such that $[\rho(o), \rho(g)] = 0$. Let $(U'; b', \rho'(o), \rho'(g))$ be the direct summand of the system $(U; b, \rho(o), \rho(g))$, where U' is the generalized λ^m -eigenspace of $\rho(o)$. Define $A' = \sqrt[m]{\rho'(o)/\lambda^m}$ by $I + \sum_{n \geq 1} C_{1/m}^n ((\rho'(o)/\lambda^m) - 1)^n$, where

$$C_{1/m}^n = \frac{1}{n!} \cdot \frac{1}{m} \left(\frac{1}{m} - 1 \right) \cdots \left(\frac{1}{m} - n + 1 \right).$$

For any real polynomial P one has $\overline{P(\rho'(o)/\lambda^m)^*} \cdot b' = b' \cdot P((\rho'(o)/\lambda^m)^{-1})$, hence $A' \in \operatorname{Aut}(U'; b')$. In particular, for any $s \in \mathbb{Z}$ one has a well-defined invariant

$$\eta_{\mathcal{J}}(\lambda; o, m; g, s) := \eta_{b'}(\rho'(g) \cdot (\lambda A')^s).$$

2.10. In the following we define the corresponding invariant for variation structures. Let \mathcal{V} be a variation structure of the group G , $\lambda \in S^1$, $m \in \mathbb{N}^*$ and $o, g \in G$ such that $og = go$. Let H be the free abelian (multiplicative) group generated by o and g in G . Then \mathcal{V} induces

on the generalized λ^m -eigenspace U' of $\rho(o)$ an abelian variation structure $\mathcal{V}' = (U'; b', \rho', V')$ of the group H . Similarly as above, we define $A' = \sqrt[m]{\rho'(o)}/\lambda^m$.

Now assume that $\lambda^m = 1$. Then

$$\rho'^e(h) = \begin{pmatrix} I & 0 \\ V'(h) & \rho'(h) \end{pmatrix}$$

for any $h \in H$, and

$$\sqrt[m]{\rho'^e(o)} = \begin{pmatrix} I & 0 \\ B' & A' \end{pmatrix}$$

where $B' = \sum_{n \geq 1} C_{1/m}^n (\rho'(o) - 1)^{n-1} V'(o)$.

Let \tilde{H} be the free abelian group generated by \tilde{o} and g . We regard H as a subgroup of \tilde{H} by the identification $(\tilde{o})^m = o$. Then $\tilde{\rho}^e(\tilde{o}) = \sqrt[m]{\rho'^e(o)}$ and $\tilde{\rho}^e(g) = \rho'^e(g)$ define a representation $\tilde{\rho}^e: \tilde{H} \rightarrow \text{Aut}(U'^e; b'^e)$. By Lemma 2.8, $\tilde{\rho}^e$ comes from a variation structure $\mathcal{V}' = (U'; b', \tilde{\rho}, \tilde{V})$ of the group \tilde{H} , such that its restriction to H is exactly \mathcal{V}' and $\tilde{\rho}(\tilde{o}) = A'$.

The above construction is compatible with the spectral decomposition of \mathcal{V}' with respect to $\rho'(g)$. Let $\oplus_{\chi} (U'_{\chi}; b'_{\chi}, \rho'_{\chi}, V'_{\chi})$ be this spectral decomposition of \mathcal{V}' , $\oplus_{\chi} A'_{\chi}$ and $\oplus_{\chi} \tilde{V}_{\chi}$ the corresponding decompositions of A' and \tilde{V} .

For any $s \in \mathbb{Z}$ define $\eta_{\mathcal{V}'}(\lambda; o, m; g, s) = \sum_{\chi} \eta_{\mathcal{V}', \chi}$, where

$$\eta_{\mathcal{V}', \chi} = \begin{cases} (1 - 2c) \text{sign } b'_{\chi} & \text{if } \chi \lambda^s = e^{2\pi i c}, 0 < c < 1 \\ -\varepsilon \text{sign } b'_{\chi} [\rho'_{\chi}(g)(\lambda A'_{\chi})^s - \rho'_{\chi}(g)^{-1}(\lambda A'_{\chi})^{-s}] & \text{if } \chi \lambda^s = 1, \text{ but } \lambda^m \neq 1 \text{ or } \chi \neq 1 \\ -\text{sign}[1 + \rho'_{\chi}(g)^{-1}(A'_{\chi})^{-s}] \tilde{V}_{\chi}(g \tilde{o}^s) & \text{if } \lambda^s = \lambda^m = \chi = 1. \end{cases}$$

Consider also

$$\eta(\lambda; m; s) = \begin{cases} 2((sc)) & \text{if } \lambda = e^{2\pi i c} \text{ and } \lambda^m = 1 \\ 0 & \text{if } \lambda^m \neq 1. \end{cases}$$

LEMMA 2.11. *With the above notation, one has*

$$\eta_{\mathcal{V}'}(\lambda; o, m; g, s) = \eta_{\mathcal{V}'}(\lambda; o, m; g, s) + \text{sign } b \cdot \eta(\lambda; m; s).$$

The proof is left to the reader.

3. THE MAIN RESULT

3.1. Let $\phi: (\mathcal{X}, 0) \rightarrow (S, 0) \subset (\mathbb{C}^2, 0)$ be a “good representative” of an n -dimensional isolated complete intersection singularity (ICIS) with discriminant space $(\Delta, 0) \subset (\mathbb{C}^2, 0)$. Let $*$ be a base point $* \in S - \Delta$, and $G = \pi_1(S - \Delta, *)$. The cohomological information about the Milnor fiber F is concentrated in the natural map $H_c^n(F) \rightarrow H^n(F)$ (we will work only with complex coefficients). Denote $U := H_c^n(F)$. If we identify the group $H^n(F)$ with the dual $U^* = \text{Hom}_{\mathbb{C}}(U, \mathbb{C})$, then the above map can be identified with the *hermitian* intersection form $b: U \rightarrow U^*$ (i.e. $b(v, w) = \int v \wedge \bar{w}$). Therefore, b satisfies $\bar{b}^* = (-1)^n b$. The monodromy representation in the orthogonal group of b is denoted by $\rho: G \rightarrow \text{Aut}(U; b)$.

If b is *degenerate*, the isometric structure $(U; b, \rho)$ is not sufficient to determine the signature of a total space of a fibring with fibre F and monodromy representation induced by ρ . The needed supplementary information lies in the variation map $V: G \rightarrow \text{Hom}_{\mathbb{C}}(U^*, U)$ (defined by a fixed trivialization of $\phi|_{\partial \mathcal{X}}: \partial \mathcal{X} \rightarrow (\mathbb{C}^2, 0)$). It is remarkable that the system $\mathcal{V} = (U; b, \rho, V)$ constitutes an $(-1)^n$ -hermitian variation structure, in the sense of Definition 2.5 (see [1, p. 11, Ch. 2]).

3.2. Let $p:(\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ be an analytic germ so that $p = p_0^l$, where l is a positive integer and p_0 is irreducible (cf. Remark 4.14).

Our goal is to compute the equivariant signatures $\sigma_\lambda(f)$ of the hypersurface singularity $f = p \circ \phi: (\mathcal{X}, 0) \rightarrow (\mathbb{C}, 0)$.

$\sigma_\lambda(f)$ is defined as follows. The monodromy defines a spectral decomposition $(\oplus_\lambda H_{n+1}(F)_\lambda, b_\lambda)$ of the pair $(H_{n+1}(F), b)$, where F is the Milnor fiber of f , and b is the (hermitian) intersection form on $H_{n+1}(F)$. Then $\sigma_\lambda(f) := \text{sign}(i^{(n+1)^2} b_\lambda)$.

Consider the splice diagram $\Gamma(p, \Delta)$ of the multilink determined by $p^{-1}(0) \cup \Delta \subset (\mathbb{C}^2, 0)$, where the multiplicity of $p^{-1}(0)$ is l , and the other multiplicities are zero. Each node of the diagram represents a Seifert component $\Sigma(a_1, \dots, a_r; S_1, \dots, S_r)$ with multilink $(\Sigma; S_1, \dots, S_r; m_1, \dots, m_r)$. Notice that the multiplicities m_i can be determined from $\Gamma(p, \Delta)$ by the corresponding splicing conditions. (For the splice geometry, see [5].) Let M_i (resp. L_i) be the topological standard meridian (resp. longitude) of the link component S_i . In the description of the geometry of Σ , the following numbers are helpful:

$$\begin{aligned} q_i &= a_1 a_2 \dots a_r / a_i, \quad i = 1, \dots, r \\ m &= \sum_{i=1}^r m_i q_i \\ m'_i &= -\sum_{j \neq i} m_j q_j / a_i, \quad n_i = \gcd(m_i, m'_i) > 0, \quad i = 1, \dots, r. \end{aligned}$$

Consider a set of integers $(\beta_i)_i$ so that $\sum_{i=1}^r \beta_i q_i = 1$. In particular, $\beta_i q_i \equiv 1$ (modulo a_i). (Recall that a_1, \dots, a_r are pairwise coprime integers.) Let δ_i be so that $\beta_i q_i + \delta_i a_i = 1$ and $s_i = m'_i \beta_i + m_i \delta_i = (m_i - \beta_i m) / a_i$.

Recall that the quotient $\Sigma/S^1 = B$ of Σ by its free S^1 -action is an r -punctured 2-sphere.

THEOREM 3.3. *By the above notations, the equivariant signature $\sigma_\lambda(f)$ is a sum over the nodes of $\Gamma(p, \Delta)$: $\sigma_\lambda(f) = \sum \sigma_\lambda(\Sigma, \rho)$, where*

(a) *For a Seifert component $\Sigma(a_1, \dots, a_r)$ (with $m = m(\Sigma)$ as above) one has*

$$\sigma_\lambda(\Sigma, \rho) = \sigma_\lambda(\Sigma) + (-1)^n \sigma(B, \rho_\lambda)$$

where $\sigma_\lambda(\Sigma) = \text{sign } b \sum_{i=1}^r \eta(\lambda; m; s_i)$ (i.e. $\sigma_\lambda(\Sigma)$ is $\sigma(F)$ -times the contribution of the Seifert component Σ in the equivariant signature $\sigma_\lambda(p)$ of the curve singularity p); and $\sigma(B, \rho_\lambda)$ is the signature of a flat bundle, provided with a non-degenerate hermitian form, over the r -punctured 2-sphere $B = \Sigma/S^1$. The hermitian flat bundle is well-determined by the system $(U; b, \rho, V)$, the geometry of Σ , and the complex number λ (see 4.7, 4.8 and Lemma 4.9). In particular, it can be computed by Meyer's method [8] (see the end of (3.4)).

(b) Assume that the variation structure over Σ is abelian (i.e. $\rho(gh) = \rho(hg)$ and $V(gh) = V(hg)$ for any pair $g, h \in \pi_1(\Sigma)$). Then $\sigma_\lambda(\Sigma, \rho)$ depends only on the restriction of the variation structure over the tubular neighbourhoods of the link components $\{S_i\}_i$. More precisely,

$$\sigma_\lambda(\Sigma, \rho) = \sum_{i=1}^r \eta_{\lambda^i}(\lambda; L_i^{q_i} M_i^{q_i}, m; L_i^{\beta_i} M_i^{-\delta_i}, s_i). \quad (*)$$

3.4. In general, the relation $(*)$ is not true, i.e. $\sigma_\lambda(\Sigma, \rho)$ cannot be computed only by the "boundary information". In order to explain this, we present in short the basic idea which guided us to the results of this paper.

Assume that the flat bundle Γ over $(B, \partial B)$ (B as above) is given by a representation $\rho_B: \pi_1(B) \rightarrow O$, where $O = O(p, q)$ (resp. $O = Sp(2p, \mathbb{R})$). Then there exists a non-flat splitting $\Gamma = \Gamma^+ \oplus \Gamma^-$ in vector bundles on which the form is, respectively, positive and negative

definite (resp. there exists a complex structure (Γ, J) on Γ). Let c_1 be the first Chern class of the signature bundle $\text{sign}(\Gamma) = \Gamma^+ - \Gamma^-$ (resp. $\text{sign}(\Gamma) = (\Gamma, J)^* - (\Gamma, J)$). (For a suitable construction, $\text{sign}(\Gamma)$ is flat near the boundary, therefore c_1 has compact support.) Then by [3], $\sigma(B, \rho_B) = \sigma(B, \text{sign}(\Gamma))$ (here the first term is defined similarly as the signature in (4.4), and the second term is the signature of B with coefficients in the signature bundle); and by [2]

$$\sigma(B, \text{sign}(\Gamma)) = \int_B c_1 - \frac{1}{2} \eta(\partial B, \text{sign}(\Gamma))$$

where the last term is the eta-invariant associated with the signature operator of ∂B and the signature bundle $\text{sign}(\Gamma)$.

If the representation is abelian, then $\text{sign}(\Gamma)$ has a flat realization; in particular $c_1 = 0$. In this case, the above formula is equivalent essentially to the formula of (*). In general, the obstruction for the validity of (*) lies in $\int c_1$.

On the other hand, for any non-degenerate representation ρ_B , the signature $\sigma(B, \rho_B)$ can be computed essentially by the Wall's formula [21] too. We identify $(B, \partial B)$ with a two-dimensional disc (with boundary) excepting $(r - 1)$ open discs. This can be cut into $r - 1$ pieces A_1, \dots, A_{r-1} , so that the interior of any of them is diffeomorphic to the annulus $\{z \in \mathbb{C}; 1 < |z| < 2\}$. Since the signature $\sigma(A_i, \rho_B|_{A_i}) = 0$, by Wall's formula $\sigma(B, \rho_B)$ is a sum of Wall's-type correction terms (which can be explicitly computed by the representation). For the details, see [8]. (In fact, this ends the computational part of our results.) Unfortunately, this algorithm, even in simple particular cases, can be very complicated. This fact emphasizes the value of (*).

3.5. We can expect that the signature $\sigma(f) = \sum_\lambda \sigma_\lambda(f)$ has a simpler expression. Indeed, one has the following theorem.

THEOREM 3.6. *The signature $\sigma(f)$ is a sum $\sigma(f) = \sum_{\text{nodes}} \sigma(\Sigma, \rho)$ over the nodes of $\Gamma(p, \Delta)$, where*

(a) *For a Seifert component Σ , the term $\sigma(\Sigma, \rho)$ is the signature of a hermitian flat bundle over the Milnor fibre of Σ . Its representation is the natural restriction of $\rho^e: G \rightarrow \text{Aut}(U^e; b^e)$, which is provided by the variation structure \mathcal{V} as in Lemma 2.8).*

(b) *If the variation structure above Σ is abelian, then*

$$\sigma(\Sigma, \rho) = \sum_{i=1}^r n_i \cdot \eta_{\mathcal{V}}(L_i^{m_i/n_i} M_i^{m_i/n_i}). \quad (**)$$

In the sum $\sum \sigma(\Sigma, \rho)$, the terms corresponding to a common edge determined by neighbouring (abelian) nodes, cancel out.

3.7. Assume in (3.4) that the representation is symmetric and definite. Then the split $\Gamma = \Gamma^+$ (or $\Gamma = \Gamma^-$) is flat; hence $c_1 = 0$. In particular, the signature can again be recovered from the boundary informations. Our version is the following.

Addendum 3.8. *Assume that the hermitian form b of the ICIS ϕ is non-degenerate, symmetric and definite. Then $\sigma_\lambda(\Sigma, \rho)$ (resp. $\sigma(\Sigma, \rho)$), for any Seifert component Σ , can be computed by the formula (*) (resp. (**)). In particular, σ depends only on the monodromy representation over the boundary components of $p^{-1}(0)$.*

Moreover, in this case, the result is true for arbitrary p (i.e. without the restriction $p = p_0^1$).

The particular case of (branched) coverings will be discussed in (4.16).

The next section contains the proof of the theorems and the addendum. In Section 5, we present some particular cases and corollaries. The basic example is the Yomdin's series. For this, we prove a quasi-periodicity property too. In the case of variation structures given by hypersurface singularities, we relate the eta-invariants to the Hodge-theoretical invariants.

4. THE PROOF OF THEOREMS 3.3, 3.6 AND 3.8.

4.1. It is convenient to replace the variation structure $(U; b, \rho, V)$ of G with a non-degenerate representation. This can be realized geometrically as follows. Let \tilde{F} be a $2n$ -dimensional oriented real manifold with boundary $\partial\tilde{F} = -\partial F$ (∂F with inverse orientation). Using the fixed trivialization of $\phi|_{\partial\mathcal{X}}$, we construct $\mathcal{X}^e = \mathcal{X} \cup_{\partial\mathcal{X}} (\tilde{F} \times S)$ with the natural extension $\phi^e: \mathcal{X}^e \rightarrow (S, 0)$ such that $\phi^e|_{\tilde{F} \times S}$ is the second projection. We prefer to close $(F, \partial F)$ with another copy \tilde{F} of F with inverse orientation. In this case, the extended fibre is $F^e = F \cup_{\partial F} (-F)$. Consider the maps:

$\beta_1: H^n(F) \rightarrow H^n(F^e)$, induced by the map $F^e \rightarrow F$ determined by the identifications $F \rightarrow F$ and $\tilde{F} \rightarrow F$; and

$\beta_2: H^n(F^e, \tilde{F}) \rightarrow H^n(F^e)$, induced by $(F^e, \emptyset) \rightarrow (F^e, \tilde{F})$.

Then $(\beta_1, \beta_2): U^* \oplus U \rightarrow H^n(F^e)$ is an isomorphism. By this isomorphism, the non-degenerate intersection form b^e (of F^e) on $U^e = U^* \oplus U$, and the monodromy representation $\rho^e: G \rightarrow \text{Aut}(U^e; b^e)$ of G , induced by ϕ^e , is given by

$$b^e = \begin{pmatrix} 0 & (-1)^n \\ 1 & b \end{pmatrix}, \quad \text{resp. } \rho^e(g) = \begin{pmatrix} I & 0 \\ V(g) & \rho(g) \end{pmatrix} \quad \text{for any } g \in G.$$

4.2. We choose an open tubular neighbourhood T of $\Delta - \{0\}$, a base space S of ϕ , and $\delta > 0$ so that

$$p: (p^{-1}(S_\delta^1) \cap S, p^{-1}(S_\delta^1) \cap S - T) \rightarrow S_\delta^1 = \{z \in \mathbb{C} : |z| = \delta\}$$

is a topological locally trivial fibration (LTF) of pair of spaces. Denote $\mathcal{P} = p^{-1}(S_\delta^1) \cap S$, $\mathcal{P}^* = \mathcal{P} - T$, $\mathcal{Y} = \phi^{-1}(\mathcal{P})$, and $\mathcal{Y}^* = \phi^{-1}(\mathcal{P}^*)$. Then $(\mathcal{P}, \mathcal{P}^*) \rightarrow S_\delta^1$ is a LTF with fibre (F_p, F_p^*) , $(\mathcal{Y}, \mathcal{Y}^*)$ is a LTF over $(\mathcal{P}, \mathcal{P}^*)$ with fibre F and $(\mathcal{Y}, \mathcal{Y}^*) \rightarrow S_\delta^1$ is a LTF with fibre (E, E^*) . Corresponding to the extension $F^e = F \cup (-F)$, we have the corresponding spaces $(\mathcal{Y}^e, \mathcal{Y}^{*e})$ and (E^e, E^{*e}) . The corresponding geometric monodromies of the fibrations $p, p \circ \phi$, and $p \circ \phi^e$ act on F_p, E, E^e , and E^{*e} . The intersection forms on the generalized eigenspaces have the equivariant signatures $\sigma_\lambda(p), \sigma_\lambda(\phi, p), \sigma_\lambda(\phi^e, p)$ (resp. $\sigma_\lambda(\phi^{*e}, p)$). The signature of F is denoted by $\sigma(F)$.

LEMMA 4.3.

- (1) $\sigma_\lambda(\phi, p) = \sigma_\lambda(\phi^e, p) + \sigma_\lambda(p) \cdot \sigma(F)$,
- (2) $\sigma_\lambda(\phi^e, p) = \sigma_\lambda(\phi^{*e}, p)$.

Proof. (1) By Wall's theorem [21, 8]: $\sigma_\lambda(\phi^e, p) = \sigma_\lambda(\phi, p) + \sigma_\lambda(\tilde{F} \times F_p) + \sigma_\lambda(K_1, K_2, K_3)$, where K_i are kernels in $W = H_n(Z)$, for $Z = \partial F_p \times \partial F$. For simplicity, we assume that $l = 1$. Consider the natural inclusions $k_i: Z \rightarrow Y_i$ ($i = 1, 2, 3$), where $Y_1 = F_p \times \partial F$, $Y_2 = \partial F_p \times \tilde{F}$, and $Y_3 = \phi^{-1}(\partial F_p)$. Then $K_i = \ker k_{i*}$, where $k_{i*}: H_n(Z) \rightarrow H_n(Y_i)$ is the induced map. It is easy to see that $K_1 = K_2$ if $n \geq 2$, and $K_1 \cap K_2 + K_1 \cap K_3 = K_1 \cap (K_2 + K_3)$ if $n = 1$. Therefore, in both cases $\sigma(K_1, K_2, K_3) = 0$.

(2) follows by Novikov additivity and by the fact that $\sigma(E^e - E^{*e}) = 0$. (Actually, when $n \geq 2$ then $H_{n+1}(E^e - E^{*e}) = 0$; if $n = 1$ then the intersection form is trivial.)

4.4. Consider the fibration $p:\mathscr{P}^*\rightarrow S^1_\delta$ with fiber F_p^* . The representation ρ^e defines a flat bundle on \mathscr{P}^* , respectively, on a fixed fibre F_p^* . The twisted cohomology group $H^1(F_p^*,\partial F_p^*; \rho^e)$ carries a $(-1)^{n+1}$ -hermitian form

$$H^1(F_p^*,\partial F_p^*; \rho^e)\otimes H^1(F_p^*,\partial F_p^*; \rho^e)\rightarrow H^2(F_p^*,\partial F_p^*; \rho^e\otimes \rho^e)\rightarrow H^1(F_p^*,\partial F_p^*; \mathbb{C})=\mathbb{C}$$

where the first map is the cup product and the second is the coefficient map induced by b^e . Since the representation is defined on the space \mathscr{P}^* , the monodromy action of p induces an action m^e on $H^1(F_p^*,\partial F_p^*; \rho^e)$. The corresponding equivariant signatures are denoted by $\sigma_\lambda(\rho^e, p)$. By a result of Meyer [8], $\sigma_\lambda(\phi^e, p) = (-1)^n \sigma_\lambda(\rho^e, p)$.

4.5. The geometric monodromy of F_p^* can be obtained by pasting the monodromy maps on the Milnor fibres F_Σ of the Seifert components Σ along boundary circles. Therefore, by the Novikov additivity we have $\sigma_\lambda(\rho^e, p) = \sum_{\text{nodes}} \sigma_\lambda(F_\Sigma, \rho^e|F_\Sigma)$, where the last term is the equivariant signature of $(H^1(F_\Sigma, \partial F_\Sigma; \rho^e|F_\Sigma); b_\Sigma^e, m_\Sigma^e)$ (defined similarly as $\sigma_\lambda(\rho^e, p)$ in (4.4)) of the Milnor fibre of Σ with the induced representation (denoted by $\rho^e|F_\Sigma$).

Moreover, again by the Novikov additivity, we can replace in each Seifert component the boundary vertices into arrowhead vertices with zero multiplicity.

4.6. Let $\Sigma(a_1, \dots, a_r)$ be a Seifert component with arrowhead vertices only and multiplicities (m_1, \dots, m_r) . Define $q_i, m'_i, \beta_i, \delta_i$ ($i = 1, \dots, r$) and m as in (3.2). A geometric realization of Σ is

$$\Sigma = \{z \in S^{2r-1} \subset \mathbb{C}^r: \sum_{i=1}^r a_{ij} z^{a_i} = 0; j = 1, \dots, r-2; (a_{ij})_{ij} \text{ generic}\}.$$

The S^1 -action is given by $t*(z_1, \dots, z_r) = (t^{q_1} z_1, \dots, t^{q_r} z_r)$, $q:\Sigma \rightarrow \Sigma/S^1 = B$ is the quotient map, and $\varphi:\Sigma \rightarrow S^1$, $\varphi(z_1, \dots, z_r) = z_1^{m_1} \dots z_r^{m_r}/|z_1^{m_1} \dots z_r^{m_r}|$ defines the Milnor fibration with Milnor fibre F_Σ . The S^1 -action induces a regular m -fold cyclic covering $F_\Sigma \rightarrow B$. The monodromy of φ is the generator $\hat{1}$ of the covering transformation group \mathbb{Z}_m .

Set $d = \gcd(m_1, \dots, m_r)$. Then F_Σ has exactly d components, which are cyclically permuted by the monodromy. If we replace m_i by m_i/d , then $a_i, q_i, \beta_i, \delta_i$ ($i = 1, \dots, r$) remain the same, but m (resp. s_i) will change in m/d (resp. s_i/d). Analysing the formula (*) with respect to this modification (use Lemma 2.11 and (2.9)), we deduce that we can assume $d = 1$.

Set $G_\Sigma = \pi_1(\Sigma, *)$ and $H = \pi_1(F_\Sigma, *)$. Then a part of the geometry of Σ is summarized in the diagram with exact rows and columns:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \\ 0 \longrightarrow & H & \longrightarrow & \pi_1(B) & \xrightarrow{c_*} & \mathbb{Z}_m & \longrightarrow 0 \\ & \uparrow = & & \uparrow_{q_*} & & \uparrow_{1 \mapsto \widehat{1}} & \\ 0 \longrightarrow & H & \longrightarrow & G_\Sigma & \xrightarrow{\varphi_*} & \mathbb{Z} & \longrightarrow 0 \\ & & & \uparrow_{k_*} & & \uparrow_{1 \mapsto m} & \\ & & & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \\ & & & \uparrow & & \uparrow & \\ & & & 0 & & 0 & \end{array}$$

The m -covering induces the first row, the Milnor fibration the second row and the S^1 -action the first column; $o = k_*(1)$ corresponds to a generic S^1 -orbit. Moreover, since

$S^1 \rightarrow \Sigma \rightarrow B$ is topologically trivial, the exact sequence $0 \rightarrow \mathbf{Z} \rightarrow G_\Sigma \rightarrow \pi_1(B) \rightarrow 0$ splits in $G_\Sigma = \mathbf{Z} \times \pi_1(B)$. In particular, o is in the centre of G_Σ .

We will need an explicit description of the maps. A basic reference is [5]. Let T_i be a tubular neighbourhood of the special fibre S_i , and choose the topological longitude L_i and meridian M_i in T_i . The quotient T_i/S^1 can be identified with one of the boundary components $\partial_i B$ of B . Then $\varphi_*(L_i^x M_i^y) = -xm'_i + ym_i$ and $q_*(L_i^x M_i^y) = xq_i - ya_i \in \pi_1(\partial_i B) = \mathbf{Z}$. On the other hand, c_* is well-defined on the conjugacy classes $[\partial_i B]$ of G , and it is determined by these values: $c_*(\partial_i B) = \hat{s}_i$. The S^1 -orbit o in T_i is represented by $L_i^{q_i} M_i^{a_i}$.

The element $\widehat{L_i^{\beta_i} M_i^{-\delta_i}}$ is a lift of $[\partial_i B]$, i.e. $q_*(L_i^{\beta_i} M_i^{-\delta_i}) = 1_{\mathbf{Z}} \in \pi_1(\partial_i B)$. Notice that $1_{\mathbf{Z}}$ projects in $\widehat{-1}$, where $\hat{1}$ represents the monodromy transformation. Therefore, $\varphi_*(L_i^{\beta_i} M_i^{-\delta_i}) = -s_i$. Since $\sum_i s_i = (1 - \sum_i \beta_i q_i)m / \prod_j a_j = 0$, the map $s_*: \pi_1(B) \rightarrow \mathbf{Z}$ given by $s_*([\partial_i B]) = -s_i$ is well-defined.

The geometric meaning of s_* is the following. A trivialization of $S^1 \rightarrow \Sigma \rightarrow B$ is a map $r: B \rightarrow \Sigma$ such that $q \circ r = 1_B$. The relation $r(\partial_i B) = L_i^{\beta_i} M_i^{-\delta_i}$ gives the correspondence between trivializations and sets of integers $(\beta_i)_i$ with $\sum_i \beta_i q_i = 1$. If r is given, then $s_* = \varphi_* \circ r_*$ is induced by $s = \varphi \circ r$. The map $r_*: \pi_1(B) \rightarrow G_\Sigma$, $r_*(\hat{g}) = g$ is characterized by: g is the unique element so that $q_*(g) = \hat{g}$ and $\varphi(g) = s_*(\hat{g})$.

4.7. The representation $\rho^e|_\Sigma: G_\Sigma \rightarrow \text{Aut}(U^e; b^e)$ determines a flat bundle. The induced flat bundle V on F_Σ is given by the induced representation $\rho^e|_{F_\Sigma}$. The quotient map determines its direct image flat bundle $R^0(q|_{F_\Sigma})_* V$. The monodromy representation $\rho_B^e: \pi_1(B) \rightarrow \text{Aut}((U^e; b^e)^{\oplus m})$ of this bundle is the following. Set $\hat{g} \in \pi_1(B)$. Let $g = r_*(\hat{g})$ be its lift as above. Denote $n(\hat{g}) = -s_*(\hat{g}) - m \cdot \lceil -s_*(\hat{g})/m \rceil$ and $\bar{g} = go^{\lceil -s_*(\hat{g})/m \rceil}$ (where $\lceil \cdot \rceil$ is the integer part). Then

$$\rho_B^e(\hat{g})(u_0, \dots, u_{m-1}) = (\rho^e(o\bar{g})u_{m-n(\hat{g})}, \dots, \rho^e(o\bar{g})u_{m-1}, \rho^e(\bar{g})u_0, \dots, \rho^e(\bar{g})u_{m-n(\hat{g})-1}).$$

The monodromy m_B^e (which commutes with $\text{im}(\rho_B^e)$), is given by

$$m_B^e(u_0, \dots, u_{m-1}) = (\rho^e(o)u_{m-1}, u_0, \dots, u_{m-2}).$$

4.8. Using again Meyer's result now for ϕ^e and $q \circ \phi^e$, we obtain the isomorphism of the following structures:

$$(H^1(F_\Sigma, \partial F_\Sigma; \rho^e|_{F_\Sigma}); b_\Sigma^e, m_\Sigma^e) \simeq (H^1(B, \partial B; \rho_B^e); b_B^e, (m_B^e)_*).$$

The monodromy $(m_B^e)_*$ is induced by a bundle morphism m_B^e which is the identity on the base B , and commutes with ρ_B^e . Therefore, the representation splits in the orthogonal sum $\oplus_\lambda (\rho_B^e)_\lambda$ with

$$(\rho_B^e)_\lambda: \pi_1(B) \rightarrow (U^e; b^e)_\lambda^{\oplus m} := \{\text{the generalized } \lambda\text{-eigenspace of } m_B^e\}.$$

Hence we have the following Lemma.

LEMMA 4.9. *If λ is an eigenvalue of m_B^e , then $\sigma_\lambda(F_\Sigma, \rho^e|_{F_\Sigma}) = \sigma(B, (\rho_B^e)_\lambda)$. For other values of λ , both terms are zero.*

Since $\sigma_\lambda(p) = \sum_{\text{nodes}} \sigma_\lambda(\Sigma)$ [15], we finished the proof of Theorem 3.3(a) (with the notation $\rho_\lambda = (\rho_B^e)_\lambda$).

4.10. Consider the representation $(\rho_B^e)_\lambda$. Let \mathcal{J}_λ^e denotes the corresponding isometric structure on B . Notice that if $\rho^e|_\Sigma$ is abelian, then $(\rho_B^e)_\lambda$ is abelian too.

LEMMA 4.11. (Atiyah *et al.* [2] and Neumann [14]). Assume that the representation $\rho^e|\Sigma$ is abelian. Then

$$(-1)^n \sigma(B, (\rho_B^e)_\lambda) = \sum_{i=1}^r \eta_{\mathcal{S}_i}(\partial_i B).$$

Therefore

$$\sigma_\lambda(\Sigma, \rho) = \sigma_\lambda(\Sigma) + \sum_{i=1}^r \eta_{\mathcal{S}_i}(\partial_i B).$$

4.12. Let $U_{\lambda^m}^e$ be the generalized λ^m -eigenspace of $\rho^e(o)$. Then $(U^e)_\lambda^{\oplus m} = \{(x, (\lambda A)^{-1}x, \dots, (\lambda A)^{-m+1}x) : x \in U_{\lambda^m}^e\}$, where $A = \sqrt[m]{\rho^e(o)/\lambda^m}$. By this identification of $(U^e)_\lambda^{\oplus m}$ with $U_{\lambda^m}^e$, the form $(b^e)_\lambda^{\oplus m}$ corresponds to $m \cdot b_{\lambda^m}^e$. Moreover, if $g = r_*(\hat{g}) \in G_\Sigma$ is a lift of \hat{g} so that $\varphi_*(g) = s_*(\hat{g})$, then $(\rho_B^e)_\lambda(\hat{g})$ becomes $(\lambda A)^{-s_*(\hat{g})} \cdot \rho^e(g)$. Now, o is $L_i^{a_i} M_i^{q_i}$, and $[\partial_i B]$ is lifted in $l_i = L_i^{\beta_i} M_i^{-\delta_i}$ and the proof of the theorem can be finished by a straightforward verification (cf. Lemma 2.11 and (2.9)).

This ends the proof of 3.3.

4.13. The proof of the Theorem 3.6 is similar to the one of Theorem 3.3. In this non-equivariant case, $\sigma(\Sigma, \rho)$ is the signature of ρ^e over F_p^* . Since $F_p^* \cap T_i$ consists of n_i copies of the $(m_i/n_i, m'_i/n_i)$ -cable of S_i , the (b) part follows.

4.14. Remark. If p is an arbitrary germ (i.e. it does not satisfy the restriction $p = p_0^!$), then only Lemma 4.3(1) fails. More precisely, in Lemma 4.3(1) appears a Wall-type correction term which depends on the global variation structure (i.e. it can not be localized over the boundary components).

4.15. The proof of 3.8. In this case it is not necessary to replace ρ by ρ^e because ρ is non-degenerate. In particular, we do not need Lemma 4.3, therefore we can work with arbitrary p (cf. Remark 4.14). Since ρ is definite, so is $(\rho_B)_\lambda$. Now apply [2] or [14].

4.16. Coverings. Consider a normal surface singularity (X, x) and a covering $\phi : (X, x) \rightarrow (\mathbb{C}^2, 0)$, which is branched over Δ . Set $X^* = \phi^{-1}(S - \Delta)$. The exact sequence associated with the non-ramified covering is $1 \rightarrow \pi_1(X^*) \rightarrow \pi_1(S - \Delta) \xrightarrow{\iota} C \rightarrow 1$. Let $|C|$ be the cardinality of C , $U = \mathbb{C}^{|C|}$ and b the natural hermitian form $b(\sum a_i e_i, \sum b_j e_j) = \sum a_i \bar{b}_i$. Let $\rho_C : C \rightarrow \text{Aut}(U; b)$ be the regular representation of C . Then the representation $\rho : \pi_1(S - \Delta) = G \rightarrow \text{Aut}(U; b)$ is in fact the composition $\rho_C \circ \tau$. Therefore, we can apply Addendum 3.8. By the above discussion, τ and the splice geometry of $\Gamma(p, \Delta)$ determine the equivariant signatures.

5. APPLICATIONS

5.1. The interested reader can apply the main theorem in the geometric cases which appear in [11].

For example, the (equivariant) signature of a hypersurface germ with one or zero-dimensional singular locus can be determined as follows. Let f be as above. Take f_2 so that the pair $\phi = (f, f_2)$ forms an ICIS (take, for example, a generic linear function). Set $p(c, d) = c$, then $f = p \circ \phi$. By our results, $\sigma_\lambda(f)$ can be computed by the variation structure of ϕ (= polar variation structure of f) and by the splice geometry of the polar curve.

5.2. The basic object of this section is the variable term of the topological series. Let $f = p \circ \phi$ be a composed singularity such that ϕ is an ICIS and $p = p'_0$, where p_0 is an irreducible curve singularity. Let $f' = p' \circ \phi$ (where $p' = (p'_0)'$ and p'_0 is irreducible) be an element of the topological series of composed singularities belonging to f [11]. By the very definition, the splice diagram $\Gamma(p', \Delta)$ arises as splice of the diagram $\Gamma(p, \Delta)$ and of another diagram Γ . By the first part of the theorem, the variable term $\sigma_\lambda(f') - \sigma_\lambda(f)$ is a sum $\sum \sigma_\lambda(\Sigma, \rho)$ over the nodes of Γ . The representations and variations over these Seifert components are induced by the *abelian* structure over the torus along with the splicing is done. In particular, the variable term is given by the formula from the second part of the theorem. In the case of the Yomdin's series we give the explicit presentation.

5.3. Assume that f_1 is a germ with one-dimensional singular locus and the pair (f_1, f_2) is an ICIS. Let $p(c, d) = c$ and $p'(c, d) = c + d^a$. The number a is so large that $\Gamma(p', \Delta)$, as a splice of $\Gamma(p, \Delta)$ and the Seifert diagram $\Gamma = \Gamma(\Sigma; a, 1, 1)$, satisfies the semi-algebraicity condition [11]. Then the correction term $\sigma_\lambda(f_1 + f_2^a) - \sigma_\lambda(f_1)$ is given by this Seifert diagram $\Gamma(\Sigma; a, 1, 1)$ with multilink structure $(0, 1, 0)$, and the variation structure of ϕ above the splicing torus.

Let \mathcal{L} (resp. \mathcal{M}) be the longitude (resp. the meridian) of $\Delta_1 = \{c = 0\} \subset \Delta$. Then the pair \mathcal{L}, \mathcal{M} generates $\mathbb{Z}^2 = \pi_1$ (splicing torus). By the splice geometry, we can make the identifications $M_1 = \mathcal{L}$, $M_2 = 1_{\mathbb{Z}^2}$, $M_3 = \mathcal{M}$. Since $[M_i]_i$ generates $H_1(\Sigma)$, using linking number argument, we get $L_1 = \mathcal{M}$, $L_2 = \mathcal{L}\mathcal{M}^a$, $L_3 = \mathcal{L}$.

We list the needed invariants:

	m_i	a_i	q_i	m'_i	n_i	β_i	δ_i	s_i	$L_i^{\beta_i} M_i^{-\delta_i}$	$L_i^{m_i/n_i} M_i^{m'_i/n_i}$
$i = 1$	0	a	1	-1	1	1	0	-1	\mathcal{M}	\mathcal{L}^{-1}
$i = 2$	1	1	a	0	1	0	1	1	$1_{\mathbb{Z}^2}$	$\mathcal{L}\mathcal{M}^a$
$i = 3$	0	1	a	$-a$	a	0	1	0	\mathcal{M}^{-1}	\mathcal{M}^{-1}

and $m = a$, and $o = \mathcal{L}\mathcal{M}^a$.

COROLLARY 5.4. (The Yomdin's series case). (1) Let $\mathcal{V} = (U; b, \rho, V)$ be the variation structure of $\phi = (f_1, f_2)$. Let \mathcal{L} and \mathcal{M} be the longitude and the meridian of $\{c = 0\}$. Then

$$\sigma_\lambda(f_1 + f_2^a) - \sigma_\lambda(f_1) =$$
$$\eta_{\mathcal{V}}(\lambda; \mathcal{L}\mathcal{M}^a, a; \mathcal{M}, -1) + \eta_{\mathcal{V}}(\lambda; \mathcal{L}\mathcal{M}^a, a; 1_{\mathbb{Z}^2}, 1) + \eta_{\mathcal{V}}(\lambda; \mathcal{L}\mathcal{M}^a, a; \mathcal{M}^{-1}, 0).$$

(2) With the same notations,

$$\sigma(f_1 + f_2^a) - \sigma(f_1) = \eta_{\mathcal{V}}(\mathcal{L}\mathcal{M}^a) - a \cdot \eta_{\mathcal{V}}(\mathcal{M}) - \eta_{\mathcal{V}}(\mathcal{L}).$$

5.5. Above we used $\eta_{\mathcal{V}}(g^{-1}) = -\eta_{\mathcal{V}}(g)$. In our expressions we preserved the multiplicative notation of $\mathbb{Z}^2 = \pi_1$ (torus); by this $1_{\mathbb{Z}^2}$ denotes its neutral element.

EXAMPLE 5.6. Let $f_1(z_1, z_2, z_3) = z_1^2 + z_2^2$; $f_2(z_1, z_2, z_3) = z_3$. Then $(U; b) = (\mathbb{C}; 0)$, $\rho(\mathcal{M}) = 1_{\mathbb{C}}$ and $V(\mathcal{M}) = -1$, and the variation structure induced on the subgroup generated by \mathcal{L} is trivial (i.e. $\rho(\mathcal{L}) = 1_{\mathbb{C}}$ and $V(\mathcal{L}) = 0$). Then $V(\mathcal{M}^a) = -a$. Since $\sigma_\lambda(f_1) = 0$ for any λ , one has $\sigma(z_1^2 + z_2^2 + z_3^a) = \eta(\mathcal{M}^a) - a \cdot \eta(\mathcal{M}) = 1 - a$, and $\sigma_\lambda(z_1^2 + z_2^2 + z_3^a) = -1$ for any λ with $\lambda^a = 1$ but $\lambda \neq 1$.

This example emphasizes the importance of the variation map: the intersection form is zero, the monodromy representation is trivial, and the whole contribution comes from the variation map.

5.7. The invariants in the surface case. If f defines an isolated singularity, we denote by $\mu(\lambda)$ the dimension of the generalized λ -eigenspace $U_\lambda(f)$ of the monodromy (acting on H_c^n (Milnor fibre)); $\mu_\pm(\lambda)$ denotes the maximal dimension of a positive (resp. negative) definite subspace of $(U_\lambda(f); b_\lambda(f))$; $\mu_0(\lambda)$ is the dimension of $\ker b_\lambda(f)$. Set

$$\mu_0 = \sum_\lambda \mu_0(\lambda), \quad \mu_\pm = \sum_\lambda \mu_\pm(\lambda), \quad \mu = \mu_0 + \mu_+ + \mu_-.$$

Obviously, $\mu_0(\lambda) = 0$ if $\lambda \neq 1$; in particular, $\mu_0 = \mu_0(1)$.

Our theorems determine $\sigma(f_1 + f_2^a)$ and $\sigma_1(f_1 + f_2^a)$ in terms of the invariants of f_1 and \mathcal{V} . On the other hand, the invariants μ and $\mu(1)$ can be computed by the zeta-function formula (see [11, 18]) as follows:

$$\mu(f_1 + f_2^a) = -1 + e(f_1) + a \cdot \sum_j d_j \mu_j^0$$

$$\mu(1)(f_1 + f_2^a) = -1 + e_1(f_1) + \dim \ker((\mathcal{L}\mathcal{M}^a - I)^{n+1})$$

where $e(f_1)$ is the Euler-characteristic of the Milnor fiber of f_1 , $e_1(f_1)$ is the order of $t - 1$ in the zeta function of f_1 , and $\{d_j, \mu_j^0\}_j$ are the usual local invariants of $\text{Sing } f_1^{-1}(0)$ (see (5.8)).

Now assume that $f_1, f_2: (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$. Then $\mu_-(1)(f_1 + f_2^a) = 0$ (see, for example, [12]). Therefore, the invariants μ_0, μ_+ and μ_- can be computed from σ, σ_1, μ and $\mu(1)$. In particular, we obtain an expression of the geometric genus $p(f_1 + f_2^a)$ by Durfee's formula $2p = \mu_+ + \mu_0$ [4]. Moreover, either by Laufer's formula [7] $\mu = K^2 + s - \mu_0 + 12p$, or by the signature formula of Durfee [4], we obtain the behaviour of the resolution invariant $(K^2 + s)(f_1 + f_2^a)$ as well. (Here, for a given resolution, K denotes its canonical class and s is the number of irreducible exceptional divisors.)

5.8. Localization. Since the geometric obstruction (for the triviality of the family $f_1 + f_2^a$) is concentrated in the singular locus \mathcal{S} of $f_1^{-1}(0)$, it is natural to express the correction term in local data provided by the transversal singularities. In this subsection we present this localization procedure in the case of the signature.

We recall the local invariants. Let $\mathcal{S}_1, \dots, \mathcal{S}_t$ be the irreducible components of \mathcal{S} . The topological degree of the restriction $\phi|_{\mathcal{S}_i}: \mathcal{S}_i \rightarrow \Delta_1 = \{c = 0\}$ is d_i . The singular fiber $F_d = \phi^{-1}(0, d)$, $(0, d) \in S \cap \Delta_1$ has exactly $\sum_{i=1}^t d_i$ isolated singularities. If $z_i \in \mathcal{S}_i - \{0\}$, the local Milnor fiber of the ICIS $(\phi^{-1}(\phi(z_i)), z_i)$ is denoted by $(F_i, \partial F_i)$. Set $U_i = H_c^n(F_i)$ and $\mu_i^0 = \dim U_i$. The disjoint union $F_{\text{loc}} = \bigcup_{i=1}^t \bigcup_{j=1}^{d_i} F_i$ of the Milnor fibers of the singularities on F_d is invariant under the geometric monodromies $m(\mathcal{L}^x \mathcal{M}^y)$, $(x, y \in \mathbb{Z})$. The corresponding algebraic monodromies on the homology group $U_{\text{loc}} = \bigoplus_{i=1}^t \bigoplus_{j=1}^{d_i} U_i$ have the form

$$\rho_{\text{loc}}(\mathcal{M}) = \bigoplus_{i=1}^t \rho_{\text{loc}, i}(\mathcal{M}), \quad \rho_{\text{loc}}(\mathcal{L}) = \bigoplus_{i=1}^t \rho_{\text{loc}, i}(\mathcal{L})$$

$$\rho_{\text{loc}, i}(\mathcal{M}) = \begin{pmatrix} M_i & & 0 \\ & \ddots & \\ 0 & & M_i \end{pmatrix}, \quad \rho_{\text{loc}, i}(\mathcal{L}) = \begin{pmatrix} 0 & & L_i \\ I & & \\ & \ddots & \\ & & I & 0 \end{pmatrix}$$

where M_i (resp. L_i) are the horizontal (resp. vertical) monodromies. The horizontal monodromy $M_i: U_i \rightarrow U_i$ is, in fact, the monodromy of the transversal singularity at $z_i \in \mathcal{S}_i - \{0\}$. This latter one is an isolated singularity with $(-1)^n$ -hermitian variation structure $(U_i; b_i, M_i, V_i)$. This means that $M_i \in \text{Aut}(U_i; b_i)$ and $V_i \in \text{Hom}(U_i^*, U_i)$ satisfies $\bar{V}_i^* = (-1)^{n+1} M_i^{-1} V_i$ and $V_i b_i = M_i - I$. This can be considered a variation structure of the group \mathbb{Z} given by $\rho_i(1) = M_i$ and $V_i(1) = V_i$. Their direct sum (over $i = 1, \dots, t$; $j = 1, \dots, d_i$) is denoted by $\mathcal{V}_{\text{loc}} = (U_{\text{loc}}; b_{\text{loc}}, \rho_{\text{loc}}(\mathcal{M}), V_{\text{loc}})$.

We can construct a representative of the (global) geometric monodromy $m(\mathcal{M})$, acting on F , which is trivial in a complement of a tubular neighbourhood of $\mathcal{S} - \{0\}$. If $i_*: U_{\text{loc}} \rightarrow U$ (resp. $i^*: U^* \rightarrow U^*_{\text{loc}}$) are induced by the natural inclusion $i: F_{\text{loc}} \rightarrow F$, then $V(\mathcal{M}) = i_* \circ V_{\text{loc}}(\mathcal{M}) \circ i^*$. The geometric monodromy of the longitude \mathcal{L} induces a non-trivial action on the boundary $D = \bigcup_{i=1}^i \bigcup_{j=1}^{d_i} \partial F_i$. This obstructs the splitting of the variation operator $V(\mathcal{L})$.

In the sequel, we replace \mathcal{V} by its restriction to the group \mathbb{Z}^2 , generated by \mathcal{M} and \mathcal{L} .

Consider the spectral decomposition $\oplus_{\chi(\mathcal{M})} (U_\chi; b_\chi, \rho_\chi, V_\chi) = \oplus_\chi \mathcal{V}_\chi$ of \mathcal{V} given by the automorphism $\rho(\mathcal{M})$; and similarly the decompositions $\mathcal{V}_{\text{loc}} = \oplus_\chi \mathcal{V}_{\text{loc}, \chi}$ and $\rho_{\text{loc}}(\mathcal{L}) = \oplus_\chi \rho_{\text{loc}, \chi}(\mathcal{L})$ given by $\rho_{\text{loc}}(\mathcal{M})$.

Assume that $\chi \neq 1$. Then $(U_\chi; b_\chi, \rho_\chi(\mathcal{M}), V_\chi(\mathcal{M})) = \mathcal{V}_{\text{loc}, \chi}$ and $\rho_\chi(\mathcal{L}) = \rho_{\text{loc}}(\mathcal{L})_\chi$. Since b_χ is non-degenerate, $V(\mathcal{L})_\chi = (\rho_{\text{loc}}(\mathcal{L}) - 1)_\chi b_{\text{loc}, \chi}^{-1}$. In particular, the structure $\mathcal{V}_{\text{loc}, \chi}$ can be extended to a structure of the group \mathbb{Z}^2 by $V_{\text{loc}}(\mathcal{L})_\chi = V(\mathcal{L})_\chi$. So, the global variation term $C_\chi(a) = \eta(\mathcal{L}\mathcal{M}^a)_\chi - a \cdot \eta(\mathcal{M})_\chi - \eta(\mathcal{L})_\chi$ is equal to the local variation term $C_{\text{loc}, \chi}(a)$ (defined by the similar formula).

Assume that $\chi = 1$. If the local transversal singularities are non-degenerate, then the variation term associated with $\chi = 1$ vanishes. In the general case, notice that the middle term of $C_1(a)$ is local: $\eta(\mathcal{M})_1 = \eta(\mathcal{V}_{\text{loc}, 1}; \mathcal{M})$. Consider the spectral decomposition $\mathcal{V}_1 = \oplus_\lambda \mathcal{V}_{1, \lambda}$ given by $\rho(\mathcal{L})$ (we do not have a local equivalent of this). Then, if $\lambda \neq 1$, then $\eta(\mathcal{V}_{1, \lambda}; \mathcal{L}\mathcal{M}^a) - \eta(\mathcal{V}_{1, \lambda}; \mathcal{L}) = 0$. So the real obstruction in the localization procedure is the term $\eta(\mathcal{V}_{1, 1}; \mathcal{L}\mathcal{M}^a) - \eta(\mathcal{V}_{1, 1}; \mathcal{L})$.

LEMMA 5.9. *Let $\mathcal{V} = (U; b, \rho, V)$ be a variation structure of the group G , and consider $g, h \in G$ so that $\rho(g) - I$ and $\rho(h) - I$ are nilpotent. Then there exists $a_0 \geq 0$ such that $\eta_{\mathcal{V}}(hg^a)$ is a constant, provided that $a > a_0$.*

Remark 5.10. Let \mathcal{V} be as above and $g \in G$ so that $\rho(g) - I$ is nilpotent and $V(g)$ is an isomorphism. Then for any $a > 0$, $\eta(g^a) = \eta(g)$. But, in Lemma 5.9, in general, a_0 cannot be equal to 0. For example, assume that $V(g)$ is an isomorphism and $\eta(g) \neq 0$. Then for any $a_0 > 0$ and $h = g^{-a_0}$, by the above remark, $\eta(hg^a) = [\text{sign}(a - a_0)] \cdot \eta(g)$.

Proof of Lemma 5.9. Let $P(z)$ be the polynomial

$$P(z) = z + \frac{z(z-1)}{2!} N + \frac{z(z-1)(z-2)}{3!} N^2 + \dots$$

where $\rho(g) = I + N$. Now, $V(hg^a) = V(h) + \rho(h)V(g^a) = V(h) + \rho(h)P(a)V(g)$. Define $V(h, g; z) = V(h) + \rho(h)P(z)V(g)$ and $SV(h, g; z) = V(h, g; z) + (-1)^{n+1} \overline{V(h, g; z)}^*$. Then, for any $m \in \mathbb{Z}$, the set $\{z | \text{sign } SV(h, g; z) = m\}$ is an algebraic constructible, in particular, there exists a_0 such that $\eta(hg^a) = \text{sign } SV(h, g; a)$ is constant, provided that $a > a_0$.

COROLLARY 5.11. *There exists a_0 so that if $a > a_0$ one has*

$$\begin{aligned} \sigma(f_1 + f_2^a) - \sigma(f_1) &= \sum_{\chi(\mathcal{M})_{\text{loc}} \neq 1} [\eta(\mathcal{V}_{\text{loc}, \chi}; \mathcal{L}\mathcal{M}^a) - a \cdot \eta(\mathcal{V}_{\text{loc}, \chi}; \mathcal{M}) - \eta(\mathcal{V}_{\text{loc}, \chi}; \mathcal{L})] \\ &\quad - a \cdot \eta(\mathcal{V}_{\text{loc}, 1}; \mathcal{M}) + C(\mathcal{V}_{1, 1}; \mathcal{L}, \mathcal{M}) \end{aligned} \tag{***}$$

where $C(\mathcal{V}_{1, 1}; \mathcal{L}, \mathcal{M}) = \lim_{a \rightarrow \infty} \eta(\mathcal{V}_{1, 1}; \mathcal{L}\mathcal{M}^a) - \eta(\mathcal{V}_{1, 1}; \mathcal{L})$ is a constant independent of a . In particular, $\sigma(f_1 + f_2^a) - \sigma(f_1 + f_2^{a_0})$ can be expressed in local terms.

If the local transversal singularities are non-degenerate, then the last two terms in the above formula vanish.

Example 5.12. The constant $C_{1,1} = C(\mathcal{V}_{1,1}; \mathcal{L}, \mathcal{M})$, in general, is not zero. Let $f_1(x, y, z) = x^3 + y^3 + \lambda xyz$ ($\lambda \neq 0$) and $f_2(x, y, z) = z$. Then $\sigma(f_1) = 0$ [6] and $\sigma(f_1 + f_2^a) = -3a + 3$. The transversal type is $3A_1$; in particular, $\rho_{\text{loc}}(\mathcal{M})$ is the identity. Therefore, on the right-hand side of (**), the sum is zero, $\eta(\mathcal{V}_{\text{loc},1}; \mathcal{M}) = 3$ and the constant $C_{1,1} = 3$.

5.13. The (quasi-)periodicity. By the monodromy theorem (i.e. the eigenvalues are roots of unity), and by Lemma 5.9, we obtain the following Corollary.

COROLLARY 5.14. $\sigma(f_1 + f_2^a)$ is a sum of a linear function and some periodic functions. The periods are given by the eigenvalues of $\rho_{\text{loc}}(\mathcal{M})$.

This corollary generalizes the corresponding result for the suspension case, conjectured by Brieskorn, Durfee and Zagier, and proved by Neumann [13].

The behaviour of the equivariant signature is even more regular. Fix λ . Consider an integer $c = c(\lambda, \mathcal{M}) > 0$ so that $\lambda^c = 1$ and the semisimple part of $\rho(\mathcal{M})$ satisfies $\rho(\mathcal{M})_{\text{ss}}^c = 1_U$. By Corollary 5.4(1), we obtain the following result.

COROLLARY 5.15. $\sigma_\lambda(f_1 + f_2^{a+nc}) = \sigma_\lambda(f_1 + f_2^a)$ for any $n \geq 0$.

Similar results are true for the equivariant Milnor numbers, too. By [11, 18] we get $\mu_\lambda(f_1 + f_2^{a+nc}) = \mu_\lambda(f_1 + f_2^a)$ for any $n \geq 0$. This shows that if the corresponding monodromies are diagonalizable, then the λ -components of the variation structure are isomorphic, provided that $\lambda \neq 1$:

$$\mathcal{V}(f_1 + f_2^{a+nc})_\lambda = \mathcal{V}(f_1 + f_2^a)_\lambda.$$

In other words, increasing a , the λ -components will not be “thicker”. The variation structure is growing by the appearance of new components which become more and more disperse. This phenomenon can be exemplified easily on the A_{a-1} -singularities $z_1^2 + z_2^2 + z_3^a$, described in Example 5.6.

We expect that this “telescopic phenomenon” is true, in general, even at the deeper level of mixed Hodge structures.

5.16. The connection with the mixed Hodge structure. Let f be an isolated singularity. Similarly as in (3.1), we construct its $(-1)^n$ -symmetric variation structure $\mathcal{V}(f) = (U; b, \rho, V)$ defined on the group \mathbf{Z} . This is essentially given by $\rho(1) = M_f$ = the algebraic monodromy, and $V(1) = V_f$ = variation map. Since V_f is an isomorphism, it determines the whole structure. In fact, it is equivalent to the real Seifert form.

The variation structures of \mathbf{Z} with $V(1)$ isomorphism are classified [12]. In the following, we recall briefly the indecomposable elements for which the eigenvalues of $\rho(1)$ are on the unit circle.

5.17 ([12]). Let $J_k: \mathbf{C}^k \rightarrow \mathbf{C}^k$ be the k -dimensional Jordan block with diagonal entries = 1.

For $\lambda \in S^1 - \{1\}$, we have, up to isomorphism, exactly two indecomposable $(-1)^n$ -hermitian variation structures with $\rho(1) = \lambda J_k$:

$$\mathcal{W}_\lambda^k(\pm 1) = (\mathbf{C}^k; b_\pm^k, \lambda J_k, (\lambda J_k - I)(b_\pm^k)^{-1})$$

where b_{\pm}^k is characterized by $(b_{\pm}^k)_{ij} = 0$ if $i + j \leq k$, and $(b_{\pm}^k)_{k,1} = \pm i^{-n^2-k+1}$. (This sign convention has Hodge theoretical motivation; in fact, it is related to the polarization formula, see (5.21)).

If $\lambda = 1$, again, there are exactly two indecomposable $(-1)^n$ -hermitian variation structures (up to isomorphism) with $\rho(1) = J_k$ and with $V = V(1)$ isomorphism. These structures are degenerate with one-dimensional $\text{Ker}(b)$. They are

$$\mathcal{W}_1^k(\pm 1) = (\mathbb{C}; 0, 1_{\mathbb{C}}, \pm i^{n^2-1}) \quad \text{if } k = 1$$

$$\mathcal{W}_1^k(\pm 1) = (\mathbb{C}^k; \tilde{b}_{\pm}^k, J_k, \tilde{V}_{\pm}^k) \quad \text{if } k \geq 2$$

where \tilde{b}_{\pm}^k is characterized by $(\tilde{b}_{\pm}^k)_{i,j} = 0$ if $i + j \leq k + 1$ and $(\tilde{b}_{\pm}^k)_{k,2} = \pm i^{-n^2-k+2}$.

5.18. An isometric structure consists of a *non-degenerate* hermitian form $(U; b)$ ($\bar{b}^* = (-1)^n b$), and a representation $\rho: G \rightarrow \text{Aut}(U; b)$. If $G = \mathbb{Z}$ then the structure $(U; b, \rho)$ is given by the pair $(b, \rho(1))$. In this case the indecomposable ones (with the same eigenvalue restriction as above) are [9]:

$$I_{\lambda}^k(\pm 1) = (\mathbb{C}^k; b_{\pm}^k, \lambda J_k) \quad \text{where } \lambda \in S^1, \quad k \geq 1.$$

Any isometric structure can be extended to a variation structure by $V(g) = (\rho(g) - I)b^{-1}$. (By this, $I_{\lambda}^k(\pm 1)$ can be identified with $\mathcal{W}_{\lambda}^k(\pm 1)$, provided that $\lambda \neq 1$.)

5.19. The following notation will be helpful: $s(\lambda) = 0$ if $\lambda \neq 1$ and $= 1$ if $\lambda = 1$. The signatures of the corresponding forms are:

$$\sigma(\mathcal{W}_{\lambda}^k(\pm 1)) = \pm \frac{1 + (-1)^{k+1+s(\lambda)}}{2}, \quad \sigma(I_{\lambda}^k(\pm 1)) = \pm \frac{1 + (-1)^{k+1}}{2}.$$

Set $\lambda = e^{2\pi ic}$ with $0 < c \leq 1$. Then, by a verification,

$$\eta(\mathcal{W}_{\lambda}^k(\pm 1); 1) = \pm (1 - 2c) \frac{1 + (-1)^{k+1}}{2}, \quad \eta(I_{\lambda}^k(\pm 1); 1) = \pm (1 - 2c) \frac{1 + (-1)^{k+1+s(\lambda)}}{2}.$$

If $\mathcal{V} = (U; b, \rho, V)$ is a variation structure of \mathbb{Z} , and $a \in \mathbb{N}^*$, then we define another structure $a^*\mathcal{V} = (U; b, a^*\rho, a^*V)$ of \mathbb{Z} by $a^*\rho(1) = \rho(a)$ and $a^*V(1) = V(a)$. By these notations, $a^*\mathcal{W}_1^k(\pm 1) = \mathcal{W}_1^k(\pm 1)$, and for $\lambda \neq 1$ one has $a^*\mathcal{W}_{\lambda}^k(\pm 1) = \mathcal{W}_{\lambda^a}^k(\pm 1)$ if $\lambda^a \neq 1$, and $a^*\mathcal{W}_{\lambda}^k(\pm 1) = I_1^k(\pm 1)$ if $\lambda^a = 1$. Therefore,

$$\eta(\mathcal{W}_{\lambda}^k(\pm 1); a) = \begin{cases} \eta(\mathcal{W}_{\lambda^a}^k(\pm 1); 1) & \text{if } \lambda = 1 \text{ or } \lambda^a \neq 1 \\ \eta(I_1^k(\pm 1); 1) & \text{if } \lambda^a = 1 \text{ and } \lambda \neq 1. \end{cases}$$

5.20. Denote by $p_{\lambda}^{p,q}(f)$ (where $r = p + q - n - s(\lambda) \geq 0$) the dimensions of the primitive spaces of the mixed Hodge structure of the germ f [19]. Consider the invariant $\Sigma pp_{\lambda, \pm}(f) = \sum (-1)^q p_{\lambda}^{p,q}$, where the sum is over the pairs (p, q) such that $r = p + q - n - s(\lambda)$ satisfies $(-1)^r = \pm 1$. Obviously, this can be derived also from the spectral pairs.

The definition of the spectral pairs is as follows.

Let $h_{\lambda}^{p,q}$ be the Hodge numbers of f ; in particular, $h_{\lambda}^{p,q} = \sum_{l \geq 0} p_{\lambda}^{p+l, q+l} (p + q \geq n + s(\lambda))$. Then the collection of the spectral pairs $Spp(f) \in \mathbb{N}[\mathbb{Q} \times \mathbb{N}]$ is

$$Spp(f) = \sum_{(\alpha, w)} h_{\exp(-2\pi i \alpha)}^{n+[-\alpha], w+s-n-[-\alpha]}(\alpha, w).$$

If we forget the weight filtration, then the information of the equivariant Hodge filtration is codified in the spectral numbers:

$$Sp(f) = \sum \alpha \in \mathbf{N}[\mathbf{Q}] \quad (\text{the sum over the spectral pairs } (\alpha, w)).$$

(For the definition of the spectral pairs and the spectral numbers, see [19, 17].)

It is remarkable that the invariant $\Sigma pp_{\lambda,+}$ is a *spectral number* invariant. Indeed, consider

$$\Sigma p_{\lambda,\pm}(f) = \# \{c | c \text{ is a spectral number with } e^{-2\pi ic} = \lambda \text{ and } (-1)^{[c]} = \pm 1\}$$

and $\Sigma p_{\lambda}(f) = \Sigma p_{\lambda,-}(f) - \Sigma p_{\lambda,+}(f)$. Then $\Sigma p_{\lambda}(f) = \Sigma pp_{\lambda,+}(f)$ (see, for example, [12]).

5.21. The connection between the Hodge structure and the variation structure is given in the relation [12]

$$\mathcal{V}(f)_{\lambda} = \bigoplus_{\substack{2n \geq p+q \geq n+s \\ r=p+q-n-s}} p_{\lambda}^{p,q}(f) \cdot \mathcal{W}_{\lambda}^{r+1}((-1)^q).$$

By (5.19) and (5.20) one has

$$\sigma_{\lambda}(f) = \begin{cases} \Sigma pp_{\lambda,+}(f) & \text{if } \lambda \neq 1 \\ \Sigma pp_{\lambda,-}(f) & \text{if } \lambda = 1. \end{cases}$$

In particular, σ_{λ} is a spectral number invariant for $\lambda \neq 1$.

In order to compute the eta-invariants, fix $\lambda = e^{-2\pi ic}$ (notice the negative sign in the exponent) with $0 \leq c < 1$. Then by (5.19) and (5.20), $\eta(\mathcal{V}(f); 1)_{\lambda} = -(1 - 2c)\Sigma pp_{\lambda,+}(f)$. On the other hand, $\eta(\mathcal{V}(f); a)_{\lambda} = \sum p_{\lambda}^{p,q} \eta(a^* \mathcal{W}_{\lambda}^{r+1}(-1)^q; 1)$ and by (5.19),

$$\eta(\mathcal{V}(f); a) = - \sum_{\substack{\lambda^a = 1 \\ \lambda \neq 1}} \Sigma pp_{\lambda,-}(f) - \sum_{\substack{\lambda^a \neq 1 \\ \text{or } \lambda = 1}} (1 - 2\{ca\})\Sigma pp_{\lambda,+}(f).$$

5.22. *The suspension case;* Let $f: (\mathbf{C}^n, 0) \rightarrow (\mathbf{C}, 0)$ be an isolated singularity. Consider $f_1: (\mathbf{C}^{n+1}, 0) \rightarrow (\mathbf{C}, 0)$ defined by $f_1(z, z_{n+1}) = f(z)$. Set $f_2 = z_{n+1}$ and $\phi = (f_1, f_2)$ as above. The singular locus of the ICIS ϕ is $\Sigma = \{z = 0\}$, and the discriminant locus Δ contains only one irreducible component $\Delta = \Delta_1$, which is smooth. In particular, $G = \mathbf{Z}$ and $\mathcal{L} = 0$. Since $\sigma(f_1) = 0$, Corollary 5.4(2), for suspensions, gives

$$\sigma(f + z_{n+1}^a) = \eta(\mathcal{V}(f); a) - a \cdot \eta(\mathcal{V}(f); 1).$$

(This can be derived also from the Sebastiani–Thom-type formula of the spectral pairs [16].)

The term $\eta(\mathcal{V}(f); 1)$ is $-\sum_{\lambda} (1 - 2c)\Sigma p_{\lambda}(f)$, which is a spectral number invariant of f . On the other hand, the term $\Sigma pp_{\lambda,-}$ in $\eta(\mathcal{V}(f); a)$ can be computed only by the spectral pairs. From this reason, if the monodromy operator of f has an even-dimensional Jordan block with eigenvalue $\lambda \neq 1$, but with $\lambda^a = 1$, then the correction term depends essentially on the block structure of the operator.

Example 5.23. Set $f_{k,l,m,n}: (\mathbf{C}^2, 0) \rightarrow (\mathbf{C}, 0)$ defined by

$$f_{k,l,m,n} = ((y - x^2)^2 - x^{5+k})((y + x^2)^2 - x^{5+l})((x - y^2)^2 - y^{5+m})((x + y^2)^2 - y^{5+n}).$$

Then the spectral numbers of $f_{-1,-1;1,1}$ and $f_{-1,1;-1,1}$ are the same, but the spectral pairs are different [17]. In the first case one has spectral pairs $(-\frac{1}{2}, 2)$ and $(\frac{1}{2}, 0)$, and in the second case $(-\frac{1}{2}, 1)$ and $(\frac{1}{2}, 1)$, whereas the other pairs are the same. Therefore, the decompositions of the variation structures (Seifert forms) are: in the first case one has

$\mathcal{W}_{-1}^2(-1)$, and in the second case $\mathcal{W}_{-1}^1(+1) \oplus \mathcal{W}_{-1}^1(-1)$, whereas the other indecomposable structures are the same. Take $a = 2$. Then $\sum_{\lambda^2=1, \lambda \neq 1} \Sigma pp_{\lambda, -}$ in the first case is $= -1$ and in the second $= 0$. In particular, $\sigma(f_{-1, -1; 1, 1} + z^2) = \sigma(f_{-1, 1; -1, 1} + z^2) + 1$.

5.24. The equivariant signature $\sigma_{\lambda}(f + z_{n+1}^a)$ ($\lambda \neq 1$) depends only on the spectrum of $f + z_{n+1}^a$ (see (5.21)), which can be computed by the spectrum of f . Therefore, $\sigma_{\lambda}(f + z_{n+1}^a)$ ($\lambda \neq 1$) is independent on the block structure of the monodromy of f . (In particular, for any a , and $\lambda \neq 1$ one has $\sigma_{\lambda}(f_{-1, -1; 1, 1} + z^a) = \sigma_{\lambda}(f_{-1, 1; -1, 1} + z^a)$.)

On the other hand, by Corollary 5.4(1), the following relation holds:

$$\sigma_1(f + z_{n+1}^a) = - \sum_{\substack{\lambda^a=1 \\ \lambda \neq 1}} \Sigma pp_{\lambda, -}(f).$$

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IMAR

Bucharest, Romania